

# Exact relationship between the entanglement entropies of XY and quantum Ising chains

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**Abstract.** – We consider two prototypical quantum models, the spin-1/2 XY chain and the quantum Ising chain and study their entanglement entropy,  $S(\ell, L)$ , of blocks of  $\ell$  spins in homogeneous or inhomogeneous systems of length  $L$ . By using two different approaches, free-fermion techniques and perturbational expansion, an exact relationship between the entropies is revealed. Using this relation we translate known results between the two models and obtain, among others, the additive constant of the entropy of the critical homogeneous quantum Ising chain and the effective central charge of the random XY chain.

**Introduction.** – Recently, we have witnessed a growing interest in entanglement effects in quantum many-body systems [1]. If an isolated quantum system is divided into two parts,  $\mathcal{A}$  and  $\mathcal{B}$ , all information about  $\mathcal{A}$  is contained in the reduced density matrix:  $\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{B}}|0\rangle\langle 0|$ , where  $|0\rangle$  denotes the ground state of the system. To quantify the quantum entanglement between  $\mathcal{A}$  and  $\mathcal{B}$ , different measures have been introduced; a frequently used quantity is the von Neumann entropy defined by:

$$S_{\mathcal{A}} = -\text{Tr}(\rho_{\mathcal{A}} \log_2 \rho_{\mathcal{A}}) = -\text{Tr}(\rho_{\mathcal{B}} \log_2 \rho_{\mathcal{B}}). \quad (1)$$

Contrary to the thermal entropy, the entanglement entropy is not an extensive quantity, but for a non-critical system it scales with the area of the interface [2] separating the block (i.e.  $\mathcal{A}$ ) and the environment ( $\mathcal{B}$ ). In the case of a one-dimensional (1d) infinite system and a finite block of length,  $\ell$ , the interface consists of a few points and the entanglement entropy generally approaches a finite value in the limit  $\ell \rightarrow \infty$ . At the critical point, however,  $S_{\mathcal{A}}$  diverges logarithmically as

$$S_{\mathcal{A}} = \frac{c}{3} \log_2 \ell + c_1, \quad (2)$$

where the prefactor,  $c$ , is universal in homogeneous systems and given by the central charge of the associated conformal field theory [3], whereas the constant,  $c_1$ , is non-universal. The relation in Eq.(2) has been generalized by Calabrese and Cardy [3] for finite systems, for different

boundary conditions, for non-critical systems in the vicinity of the transition point and for finite temperatures. In higher dimensions only a few results are available, mainly for non-interacting fermions [4] and bosons [5, 6] but also random quantum Ising models have been investigated [7].

Some of the conformal results have been tested on integrable quantum spin chains in particular on the antiferromagnetic XY-chain, defined by the Hamiltonian [8]:

$$\mathcal{H}_{XY} = \sum_{i=1}^L (J_i^x S_i^x S_{i+1}^x + J_i^y S_i^y S_{i+1}^y). \quad (3)$$

Here the  $S_i^{x,y}$ 's are spin-1/2 operators at site  $i$ ,  $S_{L+1}^{x,y} \equiv S_1^{x,y}$  and the couplings  $J_i^x$  and  $J_i^y$  may be different and site-dependent in general. In the followings, we restrict ourselves to even system sizes  $L$ . If the interaction is isotropic on average in the sense that  $[\ln J^x]_{av} = [\ln J^y]_{av}$  holds, where  $[\dots]_{av}$  stands for the average over the distribution of couplings, the model is critical, which manifests itself in the vanishing of the gap and the algebraic decay of correlation functions in the thermodynamic limit  $L \rightarrow \infty$ . For the special case of the homogeneous XX-chain, i.e. for  $J_i^x = J_i^y = J$  both the prefactor ( $c = 1$ ) and the constant  $c_1$  in Eq.(2) have been calculated exactly [10].

Another basic one-dimensional quantum model is the

quantum Ising chain (QIC) defined by the Hamiltonian

$$\mathcal{H}_I = -\frac{1}{2} \sum_{i=1}^L \lambda_i \sigma_i^x \sigma_{i+1}^x - \frac{1}{2} \sum_{i=1}^L h_i \sigma_i^z \quad (4)$$

in terms of the Pauli-operators  $\sigma_i^{x,z} = 2S_i^{x,z}$  at site  $i$  ( $\sigma_{L+1}^{x,z} \equiv \sigma_1^{x,z}$ ) and  $\lambda_i$  and  $h_i$  are nearest neighbor couplings and the transverse fields, respectively. This model exhibits a quantum phase transition if  $[\ln \lambda]_{av} = [\ln h]_{av}$ , i.e. for the homogeneous system with  $\lambda_i = \lambda$  and  $h_i = h$ , the critical point is located at  $J = h$  [9]. The saturation value of the von Neumann entropy in the homogeneous model has been found to exhibit a singularity in the vicinity of the critical point as  $S_\infty = \frac{c}{3} \log_2 \xi$ , where  $\xi$  denotes the correlation length diverging at criticality [3]. The prefactor is obtained to be  $c = 1/2$  in an analytical calculation using a mapping between the reduced density matrix of the model and the corner transfer matrix of the 2d classical Ising model [3, 11].

The entanglement entropies of the XX and the quantum Ising model have also been studied in the presence of quenched disorder, when the parameters of the models (the isotropic couplings  $J_i^x = J_i^y \equiv J_i$ , or the  $\lambda_i$  bonds and the  $h_i$  transverse fields, respectively) are independent and identically distributed random numbers. The average entropy has been calculated analytically [12] by a strong disorder renormalization group method [13], which is found to follow the logarithmic law in Eq.(2). In this case the prefactor, which is called the effective central charge and denoted by  $c_{\text{eff}}$  is obtained as  $c_{\text{eff}}(XX) = \ln 2$  for the random XX-chain and  $c_{\text{eff}}(I) = \ln 2/2$  for the random critical QIC, respectively.

Numerical studies of the entanglement entropy were performed on different XX-chains, in which the effect of a free boundary [14], a defect coupling [15], random [16] or aperiodic interactions [17] and the presence of an energy current in the system [18], etc. were investigated. For the random QIC, the location of the maxima of the entropy is used to define sample-dependent critical points [19]. The evolution of the entropy after a quench in both models is also the subject of recent investigations [20, 21].

It is known for some time that the two Hamiltonians in Eqs.(3) and (4) can be mapped into each other through a canonical transformation [22–24], which is described in the Appendix. As a consequence, the spectrum of the two Hamiltonians, as well as some correlation functions of the two models are related, as well. One might ask the question, whether a similar correspondence can be found concerning the entanglement entropies of the two models. At first thought, the existence of such a relation is not obvious since the transformation of the operators in Eqs.(34) and (35) is nonlocal, c.f. operators in  $\mathcal{A}$  for one model are expressed with operators located both in  $\mathcal{A}$  and  $\mathcal{B}$ .

In this paper, we study in detail the relation between the entanglement entropies of the two models by two approaches. In the first approach we calculate the entanglement entropies in the free-fermion representation [8].

By this method, the Hamiltonians are first expressed in terms of free fermions, which requires the solution of an eigenvalue problem of dimensions,  $L \times L$ , then after a second transformation, the systems assume the form of non-correlated fermionic modes, which is obtained by solving an  $\ell \times \ell$  eigenvalue problem. These eigenvalue problems are then compared for the two models. In the second approach, the entanglement entropy is calculated by a perturbation expansion, in terms of different powers of the coupling term connecting the two parts of the system and the expressions obtained for the two models are then compared. The relation between the entropies is then used to transfer existing results between the two models.

**Free-fermion calculation.** – The main steps of the calculations to be carried out in this section are summarized as follows. The Hamiltonians are expressed in terms of fermion operators and are diagonalized. Then, the restricted correlation matrix is constructed and it is transformed to a form corresponding to non-correlated fermionic modes by a canonical transformation. Finally, the entanglement entropy is calculated from the eigenvalues of this matrix.

*Diagonalization of the Hamiltonians.* Both models can be expressed in terms of fermion creation and annihilation operators  $c_i^+$  and  $c_i$ , respectively [8], which are obtained through the Jordan-Wigner transformation:  $a_j^\pm = S_j^x \pm iS_j^y$  and  $c_i^\pm = a_i^\pm \exp\left[\pi i \sum_{j=1}^{i-1} a_j^+ a_j^-\right]$ ,  $c_i = \exp\left[\pi i \sum_{j=1}^{i-1} a_j^+ a_j^-\right] a_i^-$ . The Hamiltonian of the XY-chain in Eq.(3) is expressed as:

$$\begin{aligned} \mathcal{H}_{XY} = & \sum_{i=1}^{L-1} \frac{1}{4} \{ (J_i^x - J_i^y) c_i^+ c_{i+1}^+ + (J_i^x + J_i^y) c_i^+ c_{i+1} + \text{h.c.} \} \\ & - \frac{1}{4} w \{ (J_L^x - J_L^y) c_L^+ c_1^+ + (J_L^x + J_L^y) c_L^+ c_1 + \text{h.c.} \} \quad (5) \end{aligned}$$

where  $w = \exp(i\pi N_c)$ , with  $N_c = \sum_{i=1}^L c_i^+ c_i$ .

Similarly one obtains for the Hamiltonian of the QIC [9] in Eq.(4):

$$\begin{aligned} \mathcal{H}_I = & \sum_{i=1}^L h_i \left( c_i^+ c_i - \frac{1}{2} \right) - \frac{1}{2} \sum_{i=1}^{L-1} \lambda_i (c_i^+ - c_i)(c_{i+1}^+ + c_{i+1}) \\ & + \frac{1}{2} w \lambda_L (c_L^+ - c_L)(c_1^+ + c_1). \quad (6) \end{aligned}$$

For both Hamiltonians, the  $N_c$  is even in the ground state, thus we have  $w = 1$ . Moreover, both Hamiltonians are quadratic in  $c_i^+$  and  $c_i$ , therefore they can be diagonalized by standard techniques [8]. In our approach, the basic quantity is a  $2L \times 2L$  matrix, denoted by  $\mathbf{T}$ , which can be interpreted as the transfer matrix of directed walks and which has been introduced in [25] for the QIC and in [24] for the XY-chain [26]. From the solution of the eigenvalue problem of  $\mathbf{T}$  one obtains both the energies of the free-fermionic modes,  $\Lambda_k > 0$ ,  $k = 1, 2, \dots, L$ , and two sets of vectors:  $\Phi_k$  and  $\Psi_k$ , both having  $L$  components [27]. For

$$\mathbf{T}_{XY} = \begin{pmatrix} 0 & 0 & J_1^y & & & -wJ_L^x & 0 \\ 0 & 0 & 0 & J_1^x & & & -wJ_L^y \\ J_1^y & 0 & 0 & 0 & J_2^x & & \\ & J_1^x & 0 & 0 & 0 & J_2^y & \\ & & J_2^x & 0 & 0 & 0 & J_3^y \\ & & & J_2^y & 0 & 0 & 0 & \ddots \\ & & & & \ddots & \ddots & \ddots & J_{L-1}^y \\ & & & & & J_{L-2}^y & 0 & 0 & 0 & J_{L-1}^x \\ -wJ_L^x & & & & & & J_{L-1}^y & 0 & 0 & 0 \\ 0 & -wJ_L^y & & & & & & J_{L-1}^x & 0 & 0 \end{pmatrix} \quad (7)$$

the XY-chain,  $\mathbf{T}$  is given by [24]:

see eq. (7) above

and the eigenvectors contain the components:  $(\Phi_k(1), \Psi_k(1), \Psi_k(2), \Phi_k(2), \Phi_k(3), \Psi_k(3), \dots, \Phi_k(L-1), \Psi_k(L-1), \Psi_k(L), \Phi_k(L))$ .

For the QIC the  $\mathbf{T}_I$  matrix reads as [25]:

$$\mathbf{T}_I = \begin{pmatrix} 0 & h_1 & & & -w\lambda_L \\ h_1 & 0 & \lambda_1 & & \\ & \lambda_1 & 0 & h_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \lambda_{L-1} & 0 & h_L \\ -w\lambda_L & & & & h_L & 0 \end{pmatrix} \quad (9)$$

and the eigenvectors have the components:  $(-\Phi_k(1), \Psi_k(1), -\Phi_k(2), \Psi_k(2), \dots, -\Phi_k(L), \Psi_k(L))$ .

As shown in [24] the  $L$  eigenvectors of  $\mathbf{T}_{XY}$  with positive eigenvalues can be divided into two classes:

i) For the first class of vectors, which we mark with odd superscripts we have  $\Phi_{2k-1}(2i) = \Psi_{2k-1}(2i-1) = 0$ , whereas the non-zero components of the vectors are obtained from the eigenvalue problem of the matrix:

$$\mathbf{T}^{(\sigma)}_I = \begin{pmatrix} 0 & J_1^y & & & -wJ_L^x \\ J_1^y & 0 & J_2^x & & \\ & J_2^x & 0 & J_3^y & \\ & & \ddots & \ddots & \ddots \\ & & & J_{L-2}^x & 0 & J_{L-1}^y \\ -wJ_L^x & & & & J_{L-1}^y & 0 \end{pmatrix} \quad (10)$$

and the eigenvectors have the components:  $(\Phi_k(1), \Psi_k(2), \Phi_k(3), \Psi_k(4), \dots, \Phi_k(L-1), \Psi_k(L))$ .

This is just the  $\mathbf{T}$  matrix of a QIC with the Hamiltonian  $H_I(\sigma)$  given in Eq.(36). Denoting the components of the vectors corresponding to this QIC by  $\Phi_k^{(\sigma)}(i)$  and  $\Psi_k^{(\sigma)}(i)$  we have the correspondences:

$$\Phi_{2k-1}(2i-1) = -\Phi_k^{(\sigma)}(i), \quad \Psi_{2k-1}(2i) = \Psi_k^{(\sigma)}(i). \quad (11)$$

ii) For the eigenvectors of the second class, which are marked with even superscripts, the vanishing components

are  $\Phi_{2k}(2i-1) = \Psi_{2k}(2i) = 0$ , whereas the non-zero components of the vectors are obtained from the eigenvalue problem of the matrix:

$$\mathbf{T}^{(\tau)}_I = \begin{pmatrix} 0 & J_1^x & & & -wJ_L^y \\ J_1^x & 0 & J_2^y & & \\ & J_2^y & 0 & J_3^x & \\ & & \ddots & \ddots & \ddots \\ & & & J_{L-2}^y & 0 & J_{L-1}^x \\ -wJ_L^y & & & & J_{L-1}^x & 0 \end{pmatrix} \quad (12)$$

and the eigenvectors have the components:  $(\Psi_k(1), \Phi_k(2), \Psi_k(3), \Phi_k(4), \dots, \Psi_k(L-1), \Phi_k(L))$ .

This is again the  $\mathbf{T}$  matrix of a QIC with the Hamiltonian  $H_I(\tau)$  given in Eq.(36). Denoting the components of the vectors corresponding to this QIC by  $\Phi_k^{(\tau)}(i)$  and  $\Psi_k^{(\tau)}(i)$  we have the correspondences:

$$\Phi_{2k}(2i) = \Psi_k^{(\tau)}(i), \quad \Psi_{2k}(2i-1) = -\Phi_k^{(\tau)}(i). \quad (13)$$

Thus we conclude that the explicit solution of  $\mathbf{T}_{XY}$  requires the diagonalization of two  $\mathbf{T}$ -s of QIC-s, which is just equivalent to the mapping described in the Appendix.

*Correlation matrix and entanglement entropy.* Next we consider a block of length  $\ell$ , consisting of spins  $i = 1, 2, \dots, \ell$  and the reduced density matrix is given by:  $\rho_\ell = \text{Tr}_{L-\ell}|0\rangle\langle 0|$ . For free fermionic systems  $\rho_\ell$  can be reconstructed from the restricted correlation matrix [28,29],  $\mathbf{G}$ , the matrix-elements of which are given by:

$$G_{m,n} = \langle 0|(c_n^+ - c_n)(c_m^+ + c_m)|0\rangle \\ = -\sum_{k=1}^L \Psi_k(m)\Phi_k(n), \quad m, n = 1, 2, \dots, \ell \quad (14)$$

For the XY chain using the properties of the vectors,  $\Phi_k$  and  $\Psi_k$  in Eqs.(11) and (13) we obtain for the matrix-elements:

$$G_{2i,2j} = 0, \quad G_{2i-1,2j-1} = 0 \\ G_{2i,2j-1} = -G_{i,j}^{(\sigma)}, \quad G_{2i-1,2j} = -G_{j,i}^{(\tau)}, \quad (15)$$

where  $G_{i,j}^{(\sigma;\tau)}$  denotes the matrix-element of the correlation matrix of the QIC with Hamiltonian  $H_I(\sigma, \tau)$ . The cor-

relation matrix for even  $\ell$  is bipartite, being composed of  $2 \times 2$  matrices

$$\begin{bmatrix} 0 & -G_{j,i}^{(\tau)} \\ -G_{i,j}^{(\sigma)} & 0 \end{bmatrix}, \quad (16)$$

$i, j = 1, 2, \dots, \ell/2$ .

In order to obtain the von Neumann entropy,  $S(\ell, L)$ , one diagonalizes  $\rho_\ell$ , which is given through a canonical transformation:

$$\mu_q = \sum_{i=1}^{\ell/2} \left[ \frac{1}{2} (v_q(i) + u_q(i)) c_i + \frac{1}{2} (v_q(i) - u_q(i)) c_i^+ \right] \quad (17)$$

where the  $v_q(i)$  and  $u_q(i)$  are real and normalized:  $\sum_i v_q^2(i) = \sum_i u_q^2(i) = 1$ . In the transformed basis we have

$$\langle 0 | \mu_q \mu_p | 0 \rangle = 0, \quad \langle 0 | \mu_q^+ \mu_p | 0 \rangle = \delta_{qp} \frac{1 + \nu_q}{2}, \quad (18)$$

for  $p, q = 1, 2, \dots, \ell$ . Thus the fermionic modes are uncorrelated and the eigenvalues of  $\rho_\ell$  are the products of  $(1 \pm \nu_q)/2$ ,  $q = 1, 2, \dots, \ell$ . The entropy of the system is given by the sum of binary entropies:

$$S(\ell, L) = - \sum_{q=1}^{\ell} \left[ \frac{1 + \nu_q}{2} \log_2 \frac{1 + \nu_q}{2} + \frac{1 - \nu_q}{2} \log_2 \frac{1 - \nu_q}{2} \right] \quad (19)$$

The  $\nu_q$ -s are the solution of the equations:

$$\mathbf{G} \mathbf{u}_q = \nu_q \mathbf{v}_q, \quad \mathbf{G}^T \mathbf{v}_q = \nu_q \mathbf{u}_q, \quad (20)$$

or, equivalently, one has

$$\mathbf{G} \mathbf{G}^T \mathbf{v}_q = \nu_q^2 \mathbf{v}_q, \quad \mathbf{G}^T \mathbf{G} \mathbf{u}_q = \nu_q^2 \mathbf{u}_q. \quad (21)$$

For the XY chain with even  $\ell$ , the matrix  $\mathbf{G} \mathbf{G}^T$  is composed of  $2 \times 2$  diagonal matrices

$$\begin{bmatrix} [(\mathbf{G}^{(\sigma)})^T \mathbf{G}^{(\sigma)}]_{i,j} & 0 \\ 0 & [\mathbf{G}^{(\tau)} (\mathbf{G}^{(\tau)})^T]_{i,j} \end{bmatrix}, \quad (22)$$

thus  $\mathbf{G} \mathbf{G}^T$  can be written in a form which consists of two diagonal blocks and the eigenvalues are obtained by solving two separate eigenvalue problems of  $(\mathbf{G}^{(\sigma)})^T \mathbf{G}^{(\sigma)}$  and  $(\mathbf{G}^{(\tau)})^T \mathbf{G}^{(\tau)}$ . Now, it follows from Eq.(19) that the entanglement entropy of the XY-chain in Eq.(3) is the sum of the entanglement entropies of the two QIC-s defined in Eq.(36):

$$S^{(XY)}(\ell, L) = S^{(\sigma)}(\ell/2, L/2) + S^{(\tau)}(\ell/2, L/2). \quad (23)$$

This relation constitutes the main result of our paper.

One can make sure easily that the above result holds not only for blocks of contiguous spins but for any block composed of pairs of adjacent sites  $(2i - 1, 2i)$ . In any other case, the matrix  $\mathbf{G} \mathbf{G}^T$  is still block-diagonal and the von Neumann entropy can be written as a sum of two terms each of which depend exclusively on the parameters

of one of the decoupled QIC's, however, these terms are no longer to be interpreted as entropies of some blocks in the QIC's.

Next, we turn to the case of odd  $\ell$ , when the correlation matrix is obtained by leaving out the last row and column of the matrix with  $\ell + 1$ , thus the structure in Eq.(16) is lost. The matrix  $\mathbf{G}$  consists of non-square submatrices of size  $l \times (l + 1)$  and  $(l + 1) \times l$  of  $\mathbf{G}^{(\tau)}$  and  $\mathbf{G}^{(\sigma)}$ , respectively. Consequently, the eigenvalues of  $\mathbf{G} \mathbf{G}^T$  can not be expressed by those obtained from the two QIC-s. Then the relation in Eq.(23) is only asymptotically valid, as  $\ell \gg 1$  and the corrections are of the order of  $1/\ell$ .

**Perturbative calculation.** – In this section we split the Hamiltonian of the system as:

$$\mathcal{H} = \mathcal{H}_A + \mathcal{H}_B + \mathcal{V}, \quad (24)$$

where  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are the Hamiltonians of the free subsystems,  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, and  $\mathcal{V}$  is the interaction term, which reads for the XY-chain as:

$$\begin{aligned} \mathcal{V}_{XY} &= H(\ell) + H(L) \\ H(\ell) &= J_\ell^x S_\ell^x S_{\ell+1}^x + J_\ell^y S_\ell^y S_{\ell+1}^y \\ H(L) &= J_L^x S_L^x S_1^x + J_L^y S_L^y S_1^y. \end{aligned} \quad (25)$$

Let us denote the eigenstates of  $\mathcal{H}_A$  by  $|\varphi_i^A\rangle$  with energies  $E_i^A$ , and similarly for  $\mathcal{H}_B$  the eigenstates are  $|\varphi_k^B\rangle$  with eigenvalues  $E_k^B$ . The ground state of the total system with Hamiltonian  $\mathcal{H}$  can be expressed in terms of  $|\varphi_i^A\rangle$  and  $|\varphi_k^B\rangle$  as  $|0\rangle = \sum_i^A \sum_k^B c(i, k) (|\varphi_i^A\rangle \otimes |\varphi_k^B\rangle)$ , so that the reduced density matrix,  $\rho_\ell$ , has the matrix-elements:

$$\begin{aligned} \langle \varphi_i^A | \rho_\ell | \varphi_j^A \rangle &= \sum_k^B \langle \langle \varphi_i^A | \otimes \langle \varphi_k^B | \rangle \mathcal{V} (| \varphi_0^B \rangle \otimes | \varphi_0^A \rangle) \rangle \langle 0 | (| \varphi_k^B \rangle \otimes | \varphi_j^A \rangle) \rangle \\ &= \sum_k^B c(i, k) c^*(j, k). \end{aligned} \quad (26)$$

Here  $c(0, 0) = 1$ , otherwise the expansion coefficients  $c(i, k)$  are calculated perturbatively. In leading order we have

$$c(i, k) = - \frac{(\langle \varphi_i^A | \otimes \langle \varphi_k^B |) \mathcal{V} (| \varphi_0^B \rangle \otimes | \varphi_0^A \rangle)}{E_i^A + E_k^B - E_0^A - E_0^B} + \dots \quad (27)$$

and the higher order terms are sums of products containing factors of the form:

$$f(i, i' | k, k') = \frac{(\langle \varphi_i^A | \otimes \langle \varphi_k^B |) \mathcal{V} (| \varphi_{i'}^B \rangle \otimes | \varphi_{k'}^A \rangle)}{E_i^A + E_k^B - E_{i'}^A - E_{k'}^B}, \quad (28)$$

such that  $E_i^A + E_k^B - E_{i'}^A - E_{k'}^B \neq 0$ . For the XY-chain with the interaction term in Eq.(25) we have  $f(i, i' | k, k') = f_\ell(i, i' | k, k') + f_L(i, i' | k, k')$ , in which the first term is given by:

$$\begin{aligned} f_\ell(i, i' | k, k') &= \frac{1}{\Delta E} [J_\ell^x \langle \varphi_i^A | S_\ell^x | \varphi_{i'}^A \rangle \langle \varphi_k^B | S_{\ell+1}^x | \varphi_{k'}^B \rangle \\ &+ J_\ell^y \langle \varphi_i^A | S_\ell^y | \varphi_{i'}^A \rangle \langle \varphi_k^B | S_{\ell+1}^y | \varphi_{k'}^B \rangle] \end{aligned} \quad (29)$$

with  $\Delta E = E_i^{\mathcal{A}} + E_k^{\mathcal{B}} - E_i^{\mathcal{A}} - E_k^{\mathcal{B}}$  and there is a similar expression for  $f_L(i, i'|k, k')$ , as well. Note that the matrix-elements in these expressions are separated as the product of two matrix-elements of the surface operators in the two subsystems.

In the next step we perform the same perturbation expansion in terms of two independent QIC-s with parameters given in Eq.(36), in which case for even  $\ell$  the perturbation is located at  $\ell/2$  and  $L/2$  and given by:

$$\begin{aligned}\tilde{H}(\ell/2) &= \frac{1}{4}J_\ell^x \sigma_{\ell/2}^x \sigma_{\ell/2+1}^x + \frac{1}{4}J_\ell^y \tau_{\ell/2}^x \tau_{\ell/2+1}^x \\ \tilde{H}(L/2) &= \frac{1}{4}J_L^x \sigma_{L/2}^x \sigma_1^x + \frac{1}{4}J_L^y \tau_{L/2}^x \tau_1^x.\end{aligned}\quad (30)$$

Using the mapping in Eq.(35) we obtain  $H(\ell) = \tilde{H}(\ell/2)$  and

$$H(L) = \frac{1}{4}J_L^x \sigma_{L/2}^x \sigma_1^x \prod_{j=1}^{L/2} \tau_j^z + \frac{1}{4}J_L^y \tau_{L/2}^x \tau_1^x \prod_{j=1}^{L/2} \sigma_j^z. \quad (31)$$

For the QIC calculation we denote by  $\tilde{f}_{\ell/2}(i, i'|k, k')$  the factor, in which we use the states labeled by  $i, i'|k, k'$  in Eq.(28). Then  $\Delta E$  remains the same, as well as the matrix-elements are invariant, so that  $f_\ell(i, i'|k, k') = \tilde{f}_{\ell/2}(i, i'|k, k')$ . For the other term,  $\tilde{f}_{L/2}(i, i'|k, k')$ , the only difference, that the transformed perturbation contains also the products  $\prod_{j=1}^{L/2} \tau_j^z$  and  $\prod_{j=1}^{L/2} \sigma_j^z$ , which commute with the Hamiltonians  $H_I(\tau)$  and  $H_I(\sigma)$ , respectively, and have the eigenvalues  $p = \pm 1$ . The excited states, however, which enter into the expansion of  $c(i, k)$  have the same parity as the ground state,  $p = 1$ , thus also  $f_L(i, i'|k, k') = \tilde{f}_{L/2}(i, i'|k, k')$ .

We conclude that for even  $\ell$  the expansion coefficients  $c(i, k)$  are identical for the XY model as well as for two independent QIC-s with parameters given in Eq. (36) in all order of the perturbation expansion. Consequently, the same is true for the elements of the reduced density matrix and finally for the entanglement entropy. In this way, we have reobtained the result already calculated in Eq.(23).

If the size of the cell,  $\ell$  is odd, then the interaction term in Eq.(25) is transformed as:

$$H(\ell) = \frac{1}{4}J_\ell^x \sigma_{(\ell+1)/2}^z + \frac{1}{4}J_\ell^y \tau_{(\ell+1)/2}^z \quad (32)$$

and similarly for  $H(L)$ . These cannot be written in a separated form in terms of the  $\sigma$  and  $\tau$  operators, thus Eq.(23) is no longer valid.

**Discussion.** – In this paper we have derived an exact relation in Eq.(23) between the entanglement entropy of the XY-chain and that of the QIC. This relation is valid for a finite block of even size and holds also for inhomogeneous couplings. Since the derivation is based on a mapping between the two correlation matrices similar relations can be obtained for another measures of the entanglement, such as the Rényi entropy or the concurrence.

Before discussing the simple consequences of the relation in Eq.(23) we begin with the comb entanglement [30] of the XY-chain, when the block  $\mathcal{A}$  consists of  $\ell \leq L/2$  spins, which occupy sites having the same parity. Then, according to Eq.(15) all elements of the matrix  $\mathbf{G}$  are zero and the entropy is  $S(\ell, L) = \ell$ , so that the block is maximally entangled with the environment. In case of strongly disordered XX chains, which possess asymptotically a random singlet ground state [13], this finding can be obtained directly since singlet bonds form exclusively between spins at sites with different parities. As can be seen, the singlet-state approximation happens to give an exact result for the comb entanglement.

First we consider a homogeneous XY chain with  $J_x = (1+\gamma)$  and  $J_y = (1-\gamma)$ , so that in the equivalent decoupled QIC-s we have  $\lambda = 1$  and  $h = (1-\gamma)/(1+\gamma)$  ( $h = (1+\gamma)/(1-\gamma)$ ) for the  $\sigma$  ( $\tau$ ) chain. In the thermodynamic limit according to Eq.(23) we have:

$$S^{(XY)}(\gamma) = S^{(I)}\left(h = \frac{1-\gamma}{1+\gamma}\right) + S^{(I)}\left(h = \frac{1+\gamma}{1-\gamma}\right), \quad (33)$$

which is indeed satisfied with the known exact results [10, 11]. For the XX-chain, which corresponds to  $\gamma = 0$ , the entropy is just the double of the entropy of the critical QIC. Consequently, the central charges are related as  $c(XX) = 2c(I)$ , which is indeed observed in the exact calculations [3, 10, 11]. For the non-universal constant in Eq.(2) we have the relation  $c_1(XX) = 2[-\frac{c(I)}{3} + c_1(I)]$ . Since  $c_1(XX)$  is also exactly known [10], in this way we have obtained the exact value of  $c_1(I)$ , which has not been known in the literature.

Our next remark concerns the entropy profile of a finite and open XX-chain, i.e. with  $J_L = 0$ , which has been recently calculated [14] and a staggered behavior is observed depending on the parity of  $\ell$ . According to our result for even  $\ell$  the mapping with the QIC is perfect, whereas for odd  $\ell$  there are finite-size corrections, which scale with  $1/\ell$ . This result is in complete agreement with the observed numerical findings.

For disordered chains the relation in Eq.(23) is valid for each (set of) samples, consequently the average entropies are also related by a factor of two. If the disorder is in the XX-form, i.e.  $J_i^x = J_i^y = J_i$ , then the effective central charges satisfy:  $c_{\text{eff}}(XX) = 2c_{\text{eff}}(I)$ , which is also in agreement with the results obtained by the strong disorder renormalization group method [12]. We obtain, however, new results, if the disorder is XY-type, i.e. generally  $J_i^x \neq J_i^y$ . According to our mapping the effective central charge of random critical XY-chains is the same as for random XX-chains and given by:  $c_{\text{eff}}(XY) = \ln 2$ .

The effect of another type of inhomogeneities on the entanglement entropy have also been considered. In this respect we mention numerical studies of the XX-chain having one or two defects, connecting the two subsystems [15], or quasi-periodic, or more generally aperiodic modulation of the couplings [17]. These perturbations result in varying

effective central charges in both models which are however related through Eq.(23).

Note added in proof: After submitting this work a paper by Cardy *et al* has appeared [31] in which  $c_1(I)$  is calculated by different methods.

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**Appendix: Mapping of the XY chain to two decoupled QIC-s.** — This type of mapping in the thermodynamic limit is presented in [22] and applied for random chains in [23]. For finite chains it is described in [24].

Let us define two sets of Pauli operators,  $\sigma_i^{x,z}$  and  $\tau_i^{x,z}$ ,  $i = 1, 2, \dots, L/2$  through the spin-1/2 operators  $S_j^{x,y}$ , with  $j = 1, 2, \dots, L$  by:

$$\begin{aligned} \sigma_i^x &= \prod_{j=1}^{2i-1} (2S_j^x), & \sigma_i^z &= 4S_{2i-1}^y S_{2i}^y \\ \tau_i^x &= \prod_{j=1}^{2i-1} (2S_j^y), & \tau_i^z &= 4S_{2i-1}^x S_{2i}^x. \end{aligned} \quad (34)$$

The inverse transformations are given by:

$$\begin{aligned} 2S_{2i-1}^x &= \sigma_i^x \prod_{j=1}^{i-1} \tau_j^z, & 2S_{2i}^x &= \sigma_i^x \prod_{j=1}^i \tau_j^z \\ 2S_{2i-1}^y &= \tau_i^x \prod_{j=1}^{i-1} \sigma_j^z, & 2S_{2i}^y &= \tau_i^x \prod_{j=1}^i \sigma_j^z. \end{aligned} \quad (35)$$

In terms of these Pauli operators the Hamiltonian operator of the XY-chain with  $L$  spins in Eq.(3) can be written as the sum of two decoupled quantum Ising chains with  $L/2$  sites:

$$\begin{aligned} \mathcal{H}_{XY} &= \frac{1}{2} [\mathcal{H}_I(\sigma) + \mathcal{H}_I(\tau)] \\ \mathcal{H}_I(\sigma) &= -\frac{1}{2} \sum_{i=1}^{L/2} J_{2i}^x \sigma_i^x \sigma_{i+1}^x - \frac{1}{2} \sum_{i=1}^{L/2} J_{2i-1}^y \sigma_i^z \sigma_{i+1}^z \\ \mathcal{H}_I(\tau) &= -\frac{1}{2} \sum_{i=1}^{L/2} J_{2i}^y \tau_i^x \tau_{i+1}^x - \frac{1}{2} \sum_{i=1}^{L/2} J_{2i-1}^x \tau_i^z \tau_{i+1}^z. \end{aligned} \quad (36)$$

Here, in the last step, we have made a gauge transformation to change the sign of the right-hand side of the last two equations. Note that although  $[\mathcal{H}_I(\sigma), \mathcal{H}_I(\tau)] = 0$ , the two chains are not independent since e.g.  $\sigma_i^x$  and  $\tau_i^x$  do not commute with each other.

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